

$f(R)$ scalar-tensor cosmology by Noether symmetry

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In the framework of $f(R)$ scalar-tensor cosmology, we use the Noether symmetry approach to find the cosmological models consistent with the Noether symmetry. We obtain the functions $f(R)$ and $H(a)$, or the corresponding differential equations, according to specific choices for the scalar field potential $V(\phi)$, the Brans-Dicke function $\omega(\phi)$, some cosmological parameters, and the constants of motion.

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I. INTRODUCTION

It is known that the expansion of universe is currently undergoing a period of acceleration which is directly measured by observations such as Type Ia Supernovae [1], large scale structure [2], cosmic microwave background (CMB) radiation [3, 4], weak lensing [5], and baryon acoustic oscillations [6]. There are two common approaches to explain the current acceleration of the universe: The first one is to introduce some new cosmological components of energy sources contributing to the so called “dark energy” in the framework of general relativity (for a review on dark energy, see, e.g., [7, 8]). The second one is to generalize R (Ricci scalar) gravity to some modified gravities [9–11]. One of the most common modified gravities is $f(R)$ gravity [12–16]. This theory relaxes the hypothesis that gravitational Lagrangian has to be a linear function of R , and as a minimal extension introduces an effective action containing a generic $f(R)$ function.

On the other hand, generalized actions of a scalar field nonminimally coupled to R gravity, as a generalization of Brans-Dicke theory [17], have been extensively studied [18]. In the present paper, we intend to more generalize such theories to include $f(R)$ gravity with a scalar field nonminimally coupled to $f(R)$ gravity. Explicitly, we aim to obtain the forms of $f(R)$, appearing in such modified action, by demanding that the Lagrangian admits the desired Noether symmetry [19, 20] (for a study of the Noether symmetry in various cosmological models see [21]). We shall see that by demanding the Noether symmetry, we can either obtain the explicit forms of the function $f(R)$ or at least find the differential equations which can be solved to obtain $f(R)$.

In Sec. (II) we introduce the action of a $f(R)$ scalar-tensor theory and obtain the corresponding field equations. In Sec. (III), we introduce in general the Noether symmetry approach, and in Sec. (IV) we apply it to the $f(R)$ scalar-tensor cosmology. In Sec. (V), we obtain the forms of $f(R)$ or the differential equations for $f(R)$. Conclusions are given in Sec. (VI).

II. COSMOLOGY FROM SCALAR-TENSOR THEORIES

Let us consider the general action

$$\mathcal{A} = \int d^4x \sqrt{-g} (\phi^2 f(R) + 4\omega(\phi) g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi)), \quad (1)$$

where the scalar field ϕ is nonminimally coupled to $f(R)$, and $\omega(\phi)$ and $V(\phi)$ are respectively the Brans-Dicke parameter and the potential as generic functions of ϕ . In order to derive the cosmological equations in a FRW metric [22], one can define a canonical Lagrangian $\mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R}, \phi, \dot{\phi})$, where $\mathcal{Q} = \{a, R, \phi\}$ is the configuration space and $\mathcal{TQ} = \{a, \dot{a}, R, \dot{R}, \phi, \dot{\phi}\}$ is the related tangent bundle on which \mathcal{L} is defined, where a dot denotes derivative with respect to the cosmic time t . The variable a is the scale factor in FRW metric, and all dynamical variables a , R , and ϕ are assumed to depend just on t to restore homogeneity and isotropy. The presence of Ricci scalar in the Lagrangian needs explanation. In fact, it is assumed that R , as well as a and ϕ , is a canonical variable because it is generally used in canonical quantization of higher order gravitational theories. However, such a position seems arbitrary, since R is not independent of a and \dot{a} . Hence, one can use the method of Lagrange multipliers to set R as a constraint of the dynamics

$$\mathcal{A} = \int dt a^3 \left\{ \phi^2 f(R) + 4\dot{\phi}^2 \omega(\phi) - V(\phi) + \lambda \left[R - 6 \left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{kc^2}{a^2} \right) \right] \right\}, \quad (2)$$

where λ here is a Lagrange multiplier. The variation of action with respect to R gives $\lambda = -\phi^2 f_R$ where $f_R := \frac{df}{dR}$. Therefore, the above action can be rewritten as

$$\mathcal{A} = \int dt a^3 \left\{ \phi^2 f(R) + 4\dot{\phi}^2 \omega(\phi) - V(\phi) - \phi^2 f_R \left[R - 6 \left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{kc^2}{a^2} \right) \right] \right\}. \quad (3)$$

By integrating by parts, and neglecting a pure divergence we obtain the point-like FRW Lagrangian

$$\begin{aligned}\mathcal{L} = & a^3\phi^2(f(R) - Rf_R) - 6\phi^2a\dot{a}^2f_R \\ & - 12a^2\dot{a}\phi\dot{\phi}f_R - 6\phi^2a^2\dot{R}\dot{a}f_{RR} + 6\phi^2ka f_R \\ & + a^3\left[4\dot{\phi}^2\omega(\phi) - V(\phi)\right].\end{aligned}\quad (4)$$

The equations of motion for a , R and ϕ are obtained respectively

$$\begin{aligned}2f_{RRR}\phi^2a^2\dot{R}^2 + 2f_{RR}\left[2a\ddot{a}\phi^2 + 2\phi\dot{a}\dot{\phi}^2\dot{R}\right. \\ \left.+ 2\phi^2a^2\ddot{R}\right] + 2f_R\left[4a\dot{a}\phi\dot{\phi} + \phi^2\dot{a}^2 + 2a^2\dot{\phi}^2 + 2a^2\phi\ddot{\phi}\right. \\ \left.+ k\phi^2 - (1/2)a^2\phi^2R\right] + a^2\phi^2f(R) \\ \left.+ a^2\left[4\dot{\phi}^2\omega(\phi) - V(\phi)\right] = 0,\end{aligned}\quad (5)$$

$$R - 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k\phi^2}{a^2}\right] = 0,\quad (6)$$

$$\begin{aligned}Af_R\left[\phi Ra^2 - 6\dot{a}^2\phi - 6a\ddot{a}\phi - 6k\phi\right] - a^3\phi f(R) \\ + 4a^3\ddot{\phi}\omega(\phi) + 12a^2\dot{a}\dot{\phi}\omega(\phi) + 2a^3\dot{\phi}^2d\omega/d\phi \\ + (1/2)a^3dV/d\phi = 0.\end{aligned}\quad (7)$$

Finally the total energy $E_{\mathcal{L}}$, corresponding to the (00) Einstein equation is obtained as

$$\begin{aligned}6\phi^2a\dot{R}\dot{a}f_{RR} + f_R\left[6\phi^2\dot{a}^2 + 12a\phi\dot{a}\dot{\phi} - a^2\phi^2R\right. \\ \left.+ 6k\phi^2\right] + a^2\phi^2f(R) - 8a^2\dot{\phi}^2\omega(\phi) \\ \left.+ a^2\left[4\dot{\phi}^2\omega(\phi) - V(\phi)\right] = 0.\end{aligned}\quad (8)$$

III. NOETHER SYMMETRY APPROACH

Let $\mathcal{L}(q^i, \dot{q}^i)$ be a canonical, non degenerate point-like Lagrangian subject to

$$\frac{\partial \mathcal{L}}{\partial t} = 0, \quad \det H_{ij} \equiv \left\| \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} \right\| \neq 0, \quad (9)$$

where H_{ij} is the Hessian matrix and a dot denotes derivative with respect to the cosmic time t . The Lagrangian \mathcal{L} is generally of the form

$$\mathcal{L} = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}), \quad (10)$$

where T and V are the ‘kinetic energy’ (with positive definite quadratic form) and ‘potential energy’ respectively. The energy function associated with \mathcal{L} is defined

$$E_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L}, \quad (11)$$

which is the total energy $T + V$ as a constant of motion. Since our cosmological problem has a finite number of

degrees of freedom, we consider only point transformations.

Any invertible transformation of the generalized positions $Q^i = Q^i(\mathbf{q})$ induces a transformation of the generalized velocities

$$\dot{Q}^i(\mathbf{q}) = \frac{\partial Q^i}{\partial q^j} \dot{q}^j, \quad (12)$$

where the matrix $\mathcal{J} = \|\partial Q^i / \partial q^j\|$ is the Jacobian of the transformation, and it is assumed to be non-zero. On the other hand, an infinitesimal point transformation is represented by a generic vector field on Q

$$\mathbf{X} = \alpha^i(\mathbf{q}) \frac{\partial}{\partial q^i}. \quad (13)$$

The induced transformation (12) is then represented by

$$\mathbf{X}^c = \alpha^i \frac{\partial}{\partial q^i} + \left(\frac{d}{dt} \alpha^i \right) \frac{\partial}{\partial \dot{q}^i}. \quad (14)$$

The Lagrangian $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})$ is invariant under the transformation by \mathbf{X} provided that

$$L_X \mathcal{L} \equiv \alpha^i \frac{\partial \mathcal{L}}{\partial q^i} + \left(\frac{d}{dt} \alpha^i \right) \frac{\partial}{\partial \dot{q}^i} \mathcal{L} = 0, \quad (15)$$

where $L_X \mathcal{L}$ is the Lie derivative of \mathcal{L} . Let us now consider the Lagrangian \mathcal{L} and its Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} - \frac{\partial \mathcal{L}}{\partial q^j} = 0. \quad (16)$$

Contracting (16) with α^j gives

$$\alpha^j \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} \right) = \alpha^j \left(\frac{\partial \mathcal{L}}{\partial q^j} \right). \quad (17)$$

On the other hand, we can write

$$\frac{d}{dt} \left(\alpha^j \frac{\partial \mathcal{L}}{\partial \dot{q}^j} \right) = \alpha^j \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} \right) + \left(\frac{d\alpha^j}{dt} \right) \frac{\partial \mathcal{L}}{\partial \dot{q}^j}, \quad (18)$$

in which the first term in the RHS can be replaced by the RHS of (17), hence (18) results in

$$\frac{d}{dt} \left(\alpha^j \frac{\partial \mathcal{L}}{\partial \dot{q}^j} \right) = L_X \mathcal{L}. \quad (19)$$

The immediate consequence of this result is the *Noether theorem* which states: if $L_X \mathcal{L} = 0$, then the function

$$\Sigma_0 = \alpha^k \frac{\partial \mathcal{L}}{\partial \dot{q}^k}, \quad (20)$$

is a constant of motion.

IV. NOETHER SYMMETRIES IN SCALAR-TENSOR COSMOLOGY

Considering the $f(R)$ scalar-tensor cosmology, the vector field associated with the Noether symmetry is

$$\mathbf{X} = A \frac{\partial}{\partial a} + B \frac{\partial}{\partial \phi} + C \frac{\partial}{\partial R} + \dot{A} \frac{\partial}{\partial \dot{a}} + \dot{B} \frac{\partial}{\partial \dot{\phi}} + \dot{C} \frac{\partial}{\partial \dot{R}}. \quad (21)$$

Now, the Noether symmetry exists if at least one of the functions A , B or C in the equation (21) is different from zero. To investigate the existence of Noether symmetry, we should write down the equation $L_X \mathcal{L} = 0$ as the following system of differential equations

$$3A\omega(\phi) + Ba \frac{d\omega}{d\phi} - 3f_R \phi \partial_\phi A + 2\omega(\phi) a \partial_\phi B = 0, \quad (22)$$

$$f_R (A\phi + 2Ba + 2a\phi \partial_a A + 2a^2 \partial_a B) + a\phi f_{RR} (C + a\partial_a C) = 0, \quad (23)$$

$$f_R (2A\phi + Ba + a\phi \partial_a A + \phi^2 \partial_\phi A + a\phi \partial_\phi B) - \frac{2}{3} \omega(\phi) a^2 \partial_a B + a\phi f_{RR} \left(C + \frac{\phi}{2} \partial_\phi C \right) = 0, \quad (24)$$

$$f_R (2\phi \partial_R A + 2a \partial_R B) + 2f_{RR} (aB + A\phi) + a\phi (C f_{RRR} + \partial_a A f_{RR} + \partial_R C f_{RR}) = 0, \quad (25)$$

$$\phi^2 f_{RR} \partial_\phi A - \frac{4}{3} \omega(\phi) a \partial_R B + 2\phi f_R \partial_R A = 0, \quad (26)$$

$$\partial_R A = 0, \quad (27)$$

which are obtained by setting to zero the coefficients of the terms \dot{a}^2 , \dot{R}^2 , $\dot{\phi}^2$, $\dot{a}\dot{R}$, $\dot{a}\dot{\phi}$, $\dot{R}\dot{\phi}$ in $L_X \mathcal{L} = 0$. Finally, we have to satisfy the constraint

$$6k\phi^2 A f_R + 3a^2 \phi^2 A (f - R f_R) - 3a^2 V(\phi) A + 2a^3 \phi B (f - R f_R) - R f_{RR} a^3 \phi^2 C - Ba^3 \frac{dV}{d\phi} + 12ka\phi f_R B + 6ka\phi^2 f_{RR} C = 0. \quad (28)$$

A solution of (22)-(27) exists if explicit forms of A , B and C are found. By using Eq. (27), the equation (25) becomes

$$f_R (2a \partial_R B) + f_{RR} (2A\phi + 2aB + a\phi \partial_a A) + a\phi \partial_R (C f_{RR}) = 0, \quad (29)$$

which can be rewritten as

$$\partial_R (2aB f_R + a\phi C f_{RR}) + f_{RR} (2A\phi + a\phi \partial_a A) = 0, \quad (30)$$

and solved with respect to R as

$$2B f_R + \phi C f_{RR} = - \left(2 \frac{A\phi}{a} + \phi \partial_a A \right) f_R + h(a, \phi), \quad (31)$$

where $h(a, \phi)$ is the integration constant. From Eq.(31) we get C

$$C = \frac{1}{\phi} \left[- \left(2B + 2 \frac{A\phi}{a} + \phi \partial_a A \right) \frac{f_R}{f_{RR}} + \frac{h(a, \phi)}{f_{RR}} \right]. \quad (32)$$

Inserting C into Eq. (23) results in

$$f_R \left(A\phi - a\phi \partial_a A - \phi a^2 \frac{\partial^2 A}{\partial a^2} \right) + a(h + a\partial_a h) = 0, \quad (33)$$

which is solved for A and h as

$$A = \left(c_1 a + \frac{c_2}{a} \right) g(\phi) \quad \text{and} \quad h = \frac{\bar{c}}{a} \lambda(\phi), \quad (34)$$

where c_1 , c_2 and \bar{c} are integration constants, and $g(\phi)$ and $\lambda(\phi)$ are some generic functions of ϕ . Substituting A and h into C we obtain

$$C = - \left[\frac{2B}{\phi} + \left(3c_1 + \frac{c_2}{a^2} \right) g(\phi) \right] \frac{f_R}{f_{RR}} + \frac{\bar{c} \lambda(\phi)}{a\phi f_{RR}}. \quad (35)$$

We leave the constraint (28) as an equation to choose suitable potential $V(\phi)$ and f_R . The remaining equations governing B , $g(\phi)$, $\omega(\phi)$ and $\lambda(\phi)$ are

$$3 \left(c_1 a + \frac{c_2}{a} \right) g(\phi) \omega(\phi) + aB \frac{d\omega}{d\phi} + 2a\omega(\phi) \partial_\phi B - 3f_R \phi \left(c_1 a + \frac{c_2}{a} \right) \frac{dg}{d\phi} = 0, \quad (36)$$

$$\frac{1}{2} f_R \left(-c_1 a + \frac{c_2}{a} \right) \phi^2 \frac{dg}{d\phi} + \frac{\bar{c}}{2} \left(\lambda + \phi \frac{d\lambda}{d\phi} \right) - \frac{2}{3} \omega(\phi) a^2 \partial_a B = 0, \quad (37)$$

$$\phi^2 f_{RR} \left(c_1 a + \frac{c_2}{a} \right) \frac{dg}{d\phi} - \frac{4}{3} \omega(\phi) a \partial_R B = 0. \quad (38)$$

By taking $\lambda(\phi) = \lambda_0 \phi^{-1}$ in Eq. (37) the term proportional to \bar{c} vanishes. The resulting equations (36), (37) and (38) are just consistent for constants $B(a, R, \phi) = B_0$ and $g(\phi) = g_0$. Equations (37) and (38) are satisfied by these solutions and Eq. (36) becomes

$$3 \left(c_1 a + \frac{c_2}{a} \right) g_0 \omega(\phi) + aB_0 \frac{d\omega}{d\phi} = 0. \quad (39)$$

One may set $c_2 = 0$ to convert this equation into a simple differential equation

$$\frac{d\omega}{d\phi} + \kappa^2 \omega = 0, \quad (40)$$

where $\kappa^2 = 3c_1 g_0 / B_0$.

For the choices of the constants c_1, g_0 and B_0 resulting in $\kappa^2 > 0$ we have the oscillating solutions

$$\omega(\phi) = \omega_0 \exp(\pm i\kappa\phi), \quad (41)$$

whereas for the choices leading to $\kappa^2 < 0$, we obtain exponential solutions

$$\omega(\phi) = \omega_0 \exp(\pm \kappa \phi). \quad (42)$$

Therefore, we find

$$\begin{aligned} A &= c_1 g_0 a, \\ B &= B_0, \\ C &= - \left[\frac{2B_0}{\phi} + 3c_1 g_0 \right] \frac{f_R}{f_{RR}} + \frac{\bar{c}\lambda_0}{a\phi^2 f_{RR}}. \end{aligned} \quad (43)$$

The existence of non zero quantities A, B and C accounts for the Noether symmetry provided that A, B, C, f_R and $V(\phi)$ satisfy the constraint (28). This equation may be converted to a differential equation for $f(R)$ as follows

$$\begin{aligned} f_R &= \frac{1}{12ka\phi^2 c_1 g} [3a^3 \phi^2 c_1 g_0 + 2a^3 \phi B] f \\ &- \left[3a^3 V(\phi) c_1 g_0 + \bar{c}\lambda_0 (Ra^2 - 6k) + Ba^3 \frac{dV}{d\phi} \right]. \end{aligned} \quad (44)$$

We may find the constant of motion, namely the Noether charge as

$$\begin{aligned} \Theta_0 &= A \frac{\partial \mathcal{L}}{\partial \dot{a}} + B \frac{\partial \mathcal{L}}{\partial \dot{\phi}} + C \frac{\partial \mathcal{L}}{\partial \dot{R}} \\ &= -6\phi a^2 c_1 g_0 [2f_R(\phi \dot{a}) + \phi a \dot{R} f_{RR}] \\ &- 12B_0 a^2 \dot{a} \phi f_R + 8B_0 a^3 \dot{\phi} \omega(\phi) \\ &+ 6\phi^2 a^2 \dot{a} \left[\frac{2B_0}{\phi} + 3c_1 g_0 \right] f_R \\ &- 6\bar{c}\lambda_0 a \dot{a}. \end{aligned} \quad (45)$$

This equation may be written as

$$\begin{aligned} f_{RR} \dot{R} &= - \frac{\Theta}{6a^3 \phi^2 c_1 g_0} - 2 \frac{d}{dt} \ln(a\phi) f_R \\ &+ \frac{4B_0 \dot{\phi} \omega(\phi)}{3\phi^2 c_1 g_0} + 3f_R H - \frac{\bar{c}\lambda_0 H}{c_1 g_0 a \phi^2}. \end{aligned} \quad (46)$$

The Friedmann equation is obtained by construction of the zero Hamiltonian constraint as

$$\begin{aligned} H &= \dot{a} \frac{\partial \mathcal{L}}{\partial \dot{a}} + \dot{R} \frac{\partial \mathcal{L}}{\partial \dot{R}} + \dot{\phi} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \\ &= f + 6f_{RR} \dot{R} H + 6f_R H^2 + 12f_R H \left(\frac{\dot{\phi}}{\phi} \right) \\ &- 4 \left(\frac{\dot{\phi}}{\phi} \right)^2 \omega(\phi) - f_R \left(R - \frac{6k}{a^2} \right) - \frac{V(\phi)}{\phi^2} = 0, \end{aligned} \quad (47)$$

or

$$\begin{aligned} H &= f - \frac{\Theta H}{\phi^2 a^3 c_1 g_0} + 12f_R H^2 \\ &- \frac{6\bar{c}\lambda_0 H^2}{c_1 a \phi^2 g_0} - f_R \left(R - \frac{6k}{a^2} \right) \\ &- \frac{V(\phi)}{\phi^2} + \frac{\dot{\phi}}{\phi^2} \omega(\phi) \left(\frac{8HB_0}{c_1 g_0} - 4\dot{\phi} \right) = 0, \end{aligned} \quad (48)$$

where we have used of (47) and that

$$\frac{\dot{\phi}}{\phi} = \frac{d}{dt} \ln(\phi), \quad \frac{d}{dt} \ln\left(\frac{a\phi}{\phi}\right) = \frac{\dot{a}}{a} = H.$$

V. $f(R)$ COSMOLOGICAL MODELS

To find some $f(R)$ scalar-tensor cosmological models consistent with the Noether symmetry, we first rewrite the constraint equation (44) as follows

$$\begin{aligned} c_1 a^2 (3a^4 \phi^2 f - 3a^4 V(\phi) - 12ka^2 \phi^2 f_R) g_0 \\ + Ba^6 (2\phi f - \frac{dV}{d\phi}) = \bar{c}\lambda_0 a^3 (Ra^2 - 6k). \end{aligned} \quad (49)$$

Then, we study the different cases according to some specific choices for $\omega(\phi), V(\phi), B_0, g_0, c_1$ and Θ . In all cases, except one, we will assume $\bar{c} = 0$.

A. $B_0 = V(\phi) = 0$

In this case, Eq.(50) is reduced to

$$a^2 \phi^2 f - a^2 V(\phi) - 4k\phi^2 f_R = 0, \quad (50)$$

or

$$f_R = \frac{a^2}{4k} f. \quad (51)$$

Using (52) in the Friedmann equation (49), we obtain

$$f \left(\frac{5}{2} + \frac{3a^2 H^2}{k} - \frac{Ra^2}{4k} \right) = \frac{\Theta H}{c_1 g_0 \phi^2 a^3} + 4\omega(\phi) \left(\frac{\dot{\phi}}{\phi^2} \right)^2 \quad (52)$$

I) Imposing $\Theta = \omega(\phi) = 0$ leads to ($f \neq 0$)

$$R = 12H^2 + \frac{10k}{a^2}. \quad (53)$$

Considering the general expression for the Ricci scalar

$$R = 12H^2 + 6aHH' + \frac{6k}{a^2}, \quad (54)$$

one may find the following differential equation by equating the RHS of (54) and (55)

$$(H^2)' - \frac{4k}{3a^3} = 0, \quad (55)$$

where ' denotes derivative with respect to a . The differential equation (56) is simply solved as

$$H^2 = -\frac{2k}{3a^2} + d_1, \quad (56)$$

where d_1 is the integration constant. This equation determines the cosmological dynamics. Now, we obtain $f(R)$. To this end, we insert (57) into (54) to obtain $a(R)$.

Then, we calculate f_R by using $a(R)$ and $f_R = \frac{df}{da} \frac{da}{dR}$ as follows

$$f_R = -\frac{df}{da} \frac{a^3}{4k}. \quad (58)$$

Finally, using (58) in (52) results in

$$f(a) = \frac{d_2}{a}, \quad (59)$$

where d_2 is another integration constant. Putting H^2 from (57) in (54), we obtain

$$R = \frac{2k}{a^2} + 12d_1. \quad (60)$$

Using this equation, we may transform $f(a)$ into $f(R)$ as

$$f(R) = d_2 \left(\frac{R - 12d_1}{2k} \right)^{1/2}. \quad (61)$$

This is viable for closed and open universes, $k = \pm 1$. For the flat universe, we may take $d_2 = \sqrt{2k}$ so that

$$f(R) = (R - 12d_1)^{1/2}. \quad (62)$$

II) Imposing $\Theta \neq 0, \omega(\phi) = 0$ in (53), results in

$$f = \frac{2k\Theta H}{c_1 g_0 \phi^2 a^3 (2k - 3a^3 H H')}, \quad (63)$$

where use has been made of (55). Note that $f_R = \frac{\partial f}{\partial a} \frac{da}{dR}$, so we may calculate separately $\frac{\partial f}{\partial a}$ and $\frac{da}{dR}$ using (63) and (55), respectively. Then, (52) casts into a differential equation for H^2 as

$$(H^2)'' = (H^2)' \left(\frac{2k}{3H^2 a^3} - \frac{5}{a} \right) - \frac{16k^2}{9a^6} \frac{1}{(H^2)'} + \frac{8k}{3a^4}. \quad (64)$$

By solving this differential equation we may find $H(a)$ which determines the cosmological dynamics. Moreover, we may insert $H(a)$ into (55) to find $a(R)$. Then, we can insert both $H(a)$ and $a(R)$ into (63) to obtain $f(R)$.

$$\text{B. } B_0 = 0, V(\phi) = \frac{1}{2} m^2 \phi^2$$

In this case, Eq.(51) leads to

$$f_R = \frac{a^2}{4k} f - \frac{m^2 a^2}{8k}. \quad (65)$$

By inserting f_R from (65) into the Friedmann equation (48), we obtain

$$(10k + 12a^2 H^2 - a^2 R) f = \frac{4k\Theta H}{c_1 g_0 \phi^2 a^3} + 5m^2 k + 2m^2 a^2 (3H^2 - R/4) + 16k\omega(\phi) \left(\frac{\dot{\phi}}{\phi^2} \right)^2. \quad (66)$$

I) Imposing $\Theta = \omega(\phi) = 0$ in (66) leads to

$$f = \frac{5k + 6a^2 H^2 - a^2 R/2}{10k + 12a^2 H^2 - a^2 R} m^2. \quad (67)$$

By inserting R from (55) into (67) we obtain

$$f = \frac{1}{2} m^2. \quad (68)$$

Obviously, this case is not physically viable because it leads to a constant f with no cosmological solutions.

II) Imposing $\Theta \neq 0, \omega(\phi) = 0$ in (66), and inserting R from (55) results in

$$f = \frac{1}{2} m^2 + \frac{2k\Theta H}{c_1 g_0 \phi^2 a^3 (2k - 3a^3 H H')}. \quad (69)$$

Using $f_R = \frac{\partial f}{\partial a} \frac{da}{dR}$, (55) and (69) we obtain the same differential equation for H^2 as (64). By solving this equation for $H(a)$, the cosmological dynamics is obtained. By inserting $H(a)$ into (55) we find $a(R)$, and by using $H(a)$ and $a(R)$ in (69) we obtain $f(R, \phi)$. Now, in order to remove ϕ in favor of R within $f(R, \phi)$, we first rewrite Eq.(22) as follows

$$3c_1 g_0 a \omega(\phi) + B_0 a \frac{d\omega}{d\phi} = 0, \quad (70)$$

where we have used of (43). This equation allows us to obtain ϕ in terms of a as a function $\phi(a)$. On the other hand, we have $a(R)$ from (55). Therefore, combining $\phi(a)$ and $a(R)$ we may obtain $\phi(R)$ by which we can replace ϕ in $f(R, \phi)$ in terms of R and obtain the desired $f(R)$.

$$\text{C. } B_0 \neq 0, V(\phi) = 0$$

In this case, Eq.(50) is reduced to

$$f_R = \frac{(3c_1 g_0 \phi + 2B_0) a^2}{12k c_1 g_0 \phi} f. \quad (71)$$

Inserting f_R from (71) into the Friedmann equation (48), and using the following definitions

$$\begin{cases} \alpha = 30k c_1 g_0 \phi + 12k B_0, \\ \beta = 12k c_1 g_0 \phi, \\ \xi = 36c_1 g_0 \phi + 24B_0 = 12\gamma, \\ \gamma = 3c_1 g_0 \phi + 2B_0, \\ \Delta = c_1 g_0 \phi^2, \\ \zeta = \alpha - 6k\gamma \\ \mu = 4 \left(\frac{\dot{\phi}}{\phi} \right)^2, \\ \nu = 8B_0 \dot{\phi} / c_1 g_0 \phi^2, \end{cases} \quad (72)$$

together with (55), we obtain

$$f \left(\frac{\zeta - 6\gamma a^3 H H'}{\beta} \right) = \frac{\Theta H}{\Delta a^3} + (\mu - \nu H) \omega(\phi). \quad (73)$$

I) Imposing $\Theta = 0, \omega(\phi) \neq 0$ in (73) leads to

$$f = \frac{D - EH}{G - Ma^3HH'}, \quad (74)$$

where

$$\begin{cases} D = \beta\mu\omega(\phi), \\ E = \beta\nu\omega(\phi), \\ G = \zeta = \beta, \\ M = 6\gamma. \end{cases} \quad (75)$$

Using $f_R = \frac{\partial f}{\partial a} \frac{da}{dR}$, (55), (70) and (74), we obtain a differential equation for H^2 . By solving this equation for $H(a)$ we obtain the cosmological dynamics. Inserting $H(a)$ into (55) results in $a(R)$. Then, we may use $H(a)$ and $a(R)$ in (73), use Eq.(70) and $a(R)$ to obtain $\phi(R)$, and finally obtain $f(R)$.

II) Imposing $\Theta \neq 0, \omega(\phi) \neq 0$ in (73) leads to

$$f = \frac{a^3(\bar{D} - \bar{E}H) + \bar{F}H}{a^3(\bar{G} - a^3\bar{M}HH')}, \quad (76)$$

where

$$\begin{cases} \bar{D} = \Delta\beta\mu\omega(\phi), \\ \bar{E} = \Delta\beta\nu\omega(\phi), \\ \bar{F} = \beta\Theta, \\ \bar{G} = \beta\Delta, \\ \bar{M} = 6\gamma\Delta. \end{cases} \quad (77)$$

Using $f_R = \frac{\partial f}{\partial a} \frac{da}{dR}$, (55), (70) and (76), we obtain a differential equation for H^2 . By solving this equation we obtain the cosmological dynamics $H(a)$. Moreover, similar to the previous case we may obtain $f(R)$

D. $\bar{c} \neq 0, V(\phi) = g_0 = 0$

In this case, Eq.(35) reads as

$$C = -\frac{2B}{\phi} \frac{f_R}{f_{RR}} + \frac{\bar{c}\lambda_0}{a\phi^2 f_{RR}}. \quad (78)$$

Inserting C from (78) into (28), results in

$$(2B_0a^3\phi)f = a^3B_0 \frac{dV}{d\phi} + \bar{c}\lambda_0(Ra^2 - 6k). \quad (79)$$

Unlike the previous procedure, in this case the constraint equation is not a differential equation containing f_R . To find an equation containing f_R we use (49). We evaluate $f_{RR}\dot{R}$ from (79) and put it in (48) to obtain

$$f + \frac{3\bar{c}\dot{\lambda}(\phi)}{aB_0}H - \frac{3\bar{c}\lambda(\phi)}{aB_0}H^2 + 6f_RH^2 \quad (80)$$

$$+ 12f_RH\left(\frac{\dot{\phi}}{\phi}\right) - 4\left(\frac{\dot{\phi}}{\phi}\right)^2\omega(\phi) - f_R\left(R - \frac{6k}{a^2}\right) - \frac{V(\phi)}{\phi^2} = 0.$$

Since $g_0 = 0$, we have $\omega(\phi) = 1$. Now, we insert (79) and (55) into (80) and obtain the required differential

equation containing f_R .

$$\begin{aligned} f_R [-6H^2 + 12H\left(\frac{\dot{\phi}}{\phi}\right) - 6aHH'] \\ = -\frac{3\bar{c}\lambda(\phi)}{aB_0}[H^2 + aHH' + H\frac{\dot{\lambda}}{\lambda}] \\ + 4\left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{1}{2\phi} \frac{dV}{d\phi} + \frac{V(\phi)}{\phi^2}. \end{aligned} \quad (81)$$

For simplicity we consider $V(\phi) = 0$, and obtain

$$f_R = \frac{\frac{3\bar{c}\lambda(\phi)}{aB_0}[H^2 + aHH' + H\frac{\dot{\lambda}}{\lambda}] - 4\left(\frac{\dot{\phi}}{\phi}\right)^2}{6H^2 - 12H\left(\frac{\dot{\phi}}{\phi}\right) + 6aHH'}. \quad (82)$$

This equation may be solved for $f(R)$ by using $H(a)$, $a(R)$, and $\phi(R)$. Using $f_R = \frac{\partial f}{\partial a} \frac{da}{dR}$, (55) and (79), a differential equation for H^2 is obtained as

$$\begin{aligned} (H^2)'' &= \frac{1}{Ha^3(2\alpha\frac{\dot{\phi}}{\phi} + \alpha\mu) - a^4\beta} \times \\ &[a^2(H^2)'[3\alpha(H^2) + \frac{\alpha a}{2}(H^2)' + (8\alpha\frac{\dot{\phi}}{\phi} + 5\alpha\mu)H - 5\beta a] + \\ &4\alpha a(H^2)[(H^2) - 2H\frac{\dot{\phi}}{\phi}] + 4\alpha k[\frac{(H^2)'}{2} - \frac{H^2}{a} - \mu\frac{H}{a}] + 4\beta k], \end{aligned} \quad (83)$$

where

$$\begin{cases} \alpha = \frac{3\bar{c}\lambda(\phi)}{B_0}, \\ \beta = 4\left(\frac{\dot{\phi}}{\phi}\right)^2, \\ \mu = \frac{\dot{\lambda}}{\lambda}. \end{cases}$$

This equation may be solved for $H^2(a)$ by using $\phi(a)$.

VI. CONCLUSIONS

In this paper, we have investigated the conditions for the existence of Noether symmetry in a $f(R)$ scalar-tensor theory of gravity in which the Ricci function $f(R)$, the scalar field potential $V(\phi)$ and the coupling function $\omega(\phi)$ are generally unknown. We have shown that the Noether symmetry may exist and further obtained a constraint between $f(R)$, $V(\phi)$ and $\omega(\phi)$. For specific choices of the functions $\omega(\phi), V(\phi)$, the parameters B_0, g_0, c_1 , and the constant of motion Θ , we have obtained explicitly the functions $f(R)$ and $H(a)$. For other cases, we have found the corresponding differential equations which can only be solved numerically.

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